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A REFINED PLATE BENDING FINITE ELEMENT

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INTRODUCTION

During the last decade, much research effort has been devoted to determine reliable element stiffness matrices for various shapes of plate bending finite elements. Attention has been given to triangular, rectangular, and quadrilateral elements. Recent surveys of presently available elements are given by Bell[1] and Gallagher[6]. A survey of rectangular finite elements for plate bending is given by Clough and Tocher[3]. Comparative studies have shown that rectangular elements show greater accuracy than triangular elements for the same number of degrees of freedom.

In summary, a number of rectangular and quadrilaterial finite elements for plate bending analysis are presently in use. Most elements show good convergence for displacements towards the true solution. However, the rate of convergence does differ substantially for different elements. Moreover, despite acceptable accuracy for displacements, some elements show poor accuracy for internal moments.

THE REFINED PLATE BENDING ELEMENT

Refinements in a finite element displacement approach can be achieved by a better approximation of the displacement field within an element. The basic unknowns in plate bending theory are the lateral deflection w, the two slopes θ_x and θ_y and the internal moments per unit length. For the present approach, at each node (i) of a finite element, the following generalized displacement components are introduced:

$$\{\delta_i\}^T = \langle w\theta_x\,\theta_y\,\phi_x\,\phi_y\,\phi_{xy}\rangle \tag{1}$$

in which: $w = w(x, y)$ = lateral deflection in z-direction, θ_x = slope about x-axis, θ_y = slope about y-axis, ϕ_x = curvature of plate surface in x-direction, ϕ_y = curvature of plate surface in y-direction, ϕ_{xy} = twist of plate surface.

The six degrees-of-freedom introduced at each nodal point lead to a 24-degree-of-freedom element and permit the choice of a higher order polynomial for the approximation of the displacement field. Using this improved field, it is possible to approximate the actual displacement field more closely, resulting in an improvement in accuracy and convergence. Through continuity requirements imposed on the curvature terms, the internal moments at all mesh points can be made continuous in this approach. Also, since internal moments are

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obtained simply by summing up curvature terms, these moments need not be computed separately.

In the present approach, only 24 terms of a complete sixth-order polynomial are retained, since the deflection function for w can be defined in terms of these 24 parameters only. With geometric symmetry of the element, no preferential direction exists. The terms with the highest even powers in x and y must be omitted in order to satisfy compatibility of w. Despite omitting these terms, geometric isotropy is retained. Retention of inappropriate terms would result in a singular transformation matrix.

Clearly, the chosen displacement function is of the non-conforming type. However, it is evident that the completeness criterion is satisfied, since all rigid body displacement modes, as well as all constant curvatures are included in the chosen functional representation. The displacement field is assumed as:

$$\begin{aligned} w = w(x, y) = {} & \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi^2 + \alpha_5 \xi\eta + \alpha_6 \eta^2 + \alpha_7 \xi^3 + \alpha_8 \xi^2\eta + \alpha_9 \xi\eta^2 \\ & + \alpha_{10}\eta^3 + \alpha_{11}\xi^4 + \alpha_{12}\xi^3\eta + \alpha_{13}\xi^2\eta^2 + \alpha_{14}\xi\eta^3 + \alpha_{15}\eta^4 + \alpha_{16}\xi^5 \\ & + \alpha_{17}\xi^4\eta + \alpha_{18}\xi^3\eta^2 + \alpha_{19}\xi^2\eta^3 + \alpha_{20}\xi\eta^4 + \alpha_{21}\eta^5 + \alpha_{22}\xi^5\eta + \alpha_{23}\xi^3\eta^3 \\ & + \alpha_{24}\xi\eta^5. \end{aligned} \tag{2}$$

In which the normalized coordinates are defined as follows:

$$\xi = x/a \quad \text{and} \quad \eta = y/b.$$

The constants α_i, with $i = 1, 2, \ldots, 24$ can be evaluated in the usual way by establishing compatibility of deformation at the four nodal points. The derivation of the element stiffness matrix follows standard procedures and is outlined in detail in[9]. The introduction of non-dimensionalized coordinates leads to a simple integration, and the final evaluation of the element stiffness matrix, which is of size 24×24, is performed in the digital computer.

For all common loading conditions, the equivalent concentrated nodal forces can be determined from an energy approach which is consistent with the evaluation of the element stiffness matrix.

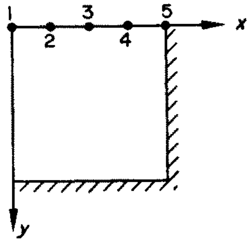
The deformed shape of a plate structure must be found in such a way that all boundary conditions adhering to a problem under consideration are fulfilled. Boundary conditions in plate bending problems usually include both force, or static, and displacement, or kinematic type. Only displacement type boundary conditions, i.e. restraints which can be expressed in terms of displacement components, can be satisfied in a displacement approach. However, due to the fact that in the present approach the three curvature terms are included in the final displacement vector, certain types of plate boundary conditions can be approximated more closely if no line moments are acting along the boundary under consideration. Finally, for the special case of a plate of abrupt change in thickness, the curvature terms should not be imposed but rather be allowed to float and come out of the solution.

ANALYSIS OF RESULTS

The results obtained for a square, isotropic plate with four simple supports discretized by four meshes having 1, 4, 16 and 64 elements per plate quadrant are shown herein. Poisson's ratio was assumed to be $\nu = 0{\cdot}30$. This simple problem was chosen in order to simplify the comparisons with other known elements and analytic solutions.

Complete deflection profiles along a center-line of the plate together with exact values, are given in Table 1 for uniformly distributed loading, and for the case of a single concentrated load. Exact values were found by evaluating the series solutions derived in[7] at all

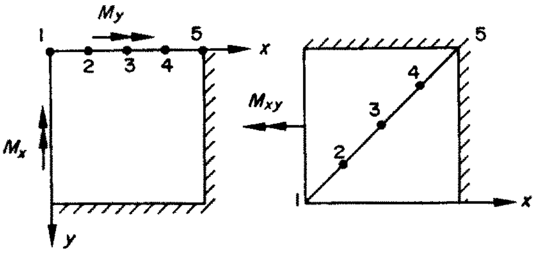
Table 1. Deflection profiles



Mesh	Point 1	Point 2	Point 3	Point 4	Point 5
(a) Uniformly loaded plate, multiplier $\frac{qL^4}{D}$					
4 × 4	0·004076		0·002948		0·
8 × 8	0·004064	0·003778	0·002939	0·001624	0·
16 × 16	0·004063	0·003776	0·002938	0·001623	0·
Exact value	0·004062	0·003776	0·002938	0·001623	0·
(b) Single concentrated load, multiplier $\frac{PL^2}{D}$					
4 × 4	0·011497		0·007144		0·
8 × 8	0·011572	0·010066	0·007141	0·003670	0·
16 × 16	0·011593	0·010068	0·007139	0·003669	0·
Exact value	0·01160	0·010066	0·007139	0·003668	0·

points of interest. Good agreement of displacements is apparent as the convergence is fast and monotonic. Table 2 lists the computed internal moments M_x, M_y and M_{xy} along a center-line of the plate, together with exact values, where available. From these results, it is evident that excellent accuracy for displacements and internal moments is obtained with the refined plate element.

Table 2. Plate moments M_x, M_y and M_{xy} uniformly loaded plate (Multiplier qL^2)

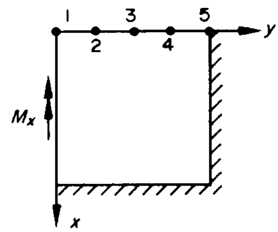


Moment	Mesh	Point 1	Point 2	Point 3	Point 4	Point 5
M_x	4 × 4	0·0454		0·0383		0·
	8 × 8	0·0475	0·0454	0·0385	0·0248	0·
	16 × 16	0·0478	0·0457	0·0388	0·0248	0·
	Exact value	0·0479	0·0458	0·0390	0·0250	0·
M_y	4 × 4	0·0454		0·0350		0·
	8 × 8	0·0475	0·0444	0·0348	0·0205	0·
	16 × 16	0·0478	0·0447	0·0355	0·0203	0·
	Exact value	0·0479	0·0448	0·0356	0·0204	0·
M_{xy}	4 × 4	0·		0·0133		0·0319
	8 × 8	0·	0·0037	0·0134	0·0252	0·0288
	16 × 16	0·	0·0038	0·0134	0·0252	0·0324
	Exact value	0·	0·0037	0·0134	0·0252	0·0324

Table 3. Effect of boundary conditions on center deflection

Boundary conditions	Mesh 2 × 2	Mesh 4 × 4	Mesh 8 × 8	Mesh 16 × 16	Multiplier
(a) Center deflection under uniformly distributed load					
Type I	0·004187	0·004076	0·004064	0·004063	
Type II	0·004066	0·004063	0·004062	0·004062	$\frac{qL^4}{D}$
Type III	0·004065	0·004063	0·004062	0·004062	
Exact value		0·004062			
(b) Center deflection under concentrated load					
Type I	0·011265	0·011497	0·011572	0·011593	
Type II	0·011184	0·011478	0·011570	0·011593	$\frac{PL^2}{D}$
Type III	0·011180	0·011478	0·011570	0·011593	
Exact value		0·01160			

Table 4. Effect of boundary conditions on plate moments M_x



Boundary conditions	Point 1	Point 2	Point 3	Point 4	Point 5	Multiplier
(a) Uniformly distributed load, Mesh 16 × 16						
Type I	0·0478	0·0457	0·0388	0·0248	−0·0010	
Type II	0·0478	0·0457	0·0387	0·0248	−0·0002	qL^2
Type III	0·0478	0·0457	0·0388	0·0248	0·	
Exact value	0·0479	0·0458	0·0390	0·0250	0·	
(b) Single concentrated load, Mesh 16 × 16						
Type I		0·1230	0·0588	0·0244	−0·0025	
Type II		0·1230	0·0588	0·0245	−0·0004	P
Type III		0·1226	0·0586	0·0242	0·	
Exact value		0·1231	0·0585	0·0251	0·	

In order to study the effect of the enforcement of boundary conditions, a number of comparisons have been made. For the purpose of these comparisons the following types of boundary conditions are defined:

Type I Only displacement type boundary conditions, associated with w, $\partial w/\partial x$ and $\partial w/\partial y$ are enforced.

Type II In addition to the constraints of Type I, curvature terms derived from a knowledge of the geometry of the deflected surface are enforced.

Type III In addition to the constraints of Type II, curvature terms derived from static considerations are enforced.

Tables 3 and 4 list, in part, the results of this investigation. Comparing the computed values for the center deflection of the problem at hand for the different types of boundary conditions enforced, it can be stated that, if boundary conditions of Types II and III are enforced, the structure tends to become stiffer. However, for finer meshes no difference can be recognized, thus leading to the conclusion that the imposition of additional curvature constraints does not improve the computed center deflection as this would be expected from the Minimum Potential Energy Theorem. For internal moments however, the imposition of additional curvature terms does improve the moment field, especially in the vicinity of the boundaries.

COMPARISON WITH EXISTING PLATE ELEMENTS

A direct comparison in terms of mesh size of the different finite elements used for this example is not appropriate, since the computational effort is different for different elements and meshes. Most results available in the literature are listed separately for each mesh size. In a finite element approach involving fine meshes, the major part of the computer time required is used for the solution of the typically large system of simultaneous equations. Hence, a more reasonable way of comparing the results is to plot the percentage error in deflection or internal moment against the number of degrees-of-freedom; the solution time being directly proportional to this number in the Cholesky decomposition technique.

In Figs. 1 and 2 the percentage error in central deflection is plotted against the number of degrees-of-freedom of the problem for some known finite elements. Clearly, the refined

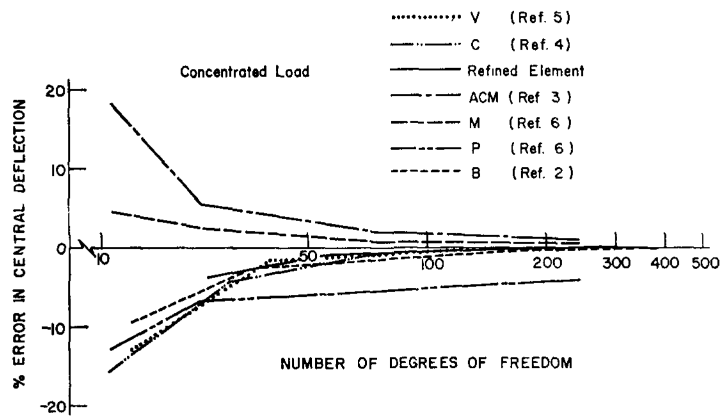


Fig. 1. Percentage error in central deflection vs number of degrees-of-freedom—simply supported plate under concentrated load.

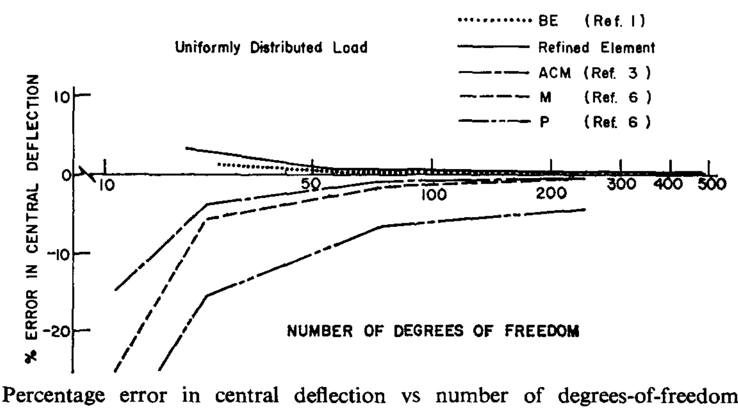


Fig. 2. Percentage error in central deflection vs number of degrees-of-freedom—simply supported plate under uniformly distributed load.

element compares favorably with most other elements for a given number of degrees-of-freedom. A better index for comparison would be the time of the total computational effort needed for the entire solution of larger sized problems. In fact, the computer time needed to generate the element stiffness matrices, to assemble the system stiffness matrix, to generate force vectors, to solve the resulting large system of simultaneous equations and finally, to find all internal moments would be a better measure for the discussion of the relative merits of different proposed elements. Comparisions for other sample problems are made in[9].

SUMMARY AND CONCLUSIONS

A refined rectangular plate element for use in a finite element analysis of elastic plates is presented. Along with the three basic nodal displacements, three curvature terms are entered as unknowns in the vector of generalized displacements. Results found for different example problems solved indicate that the refined element gives very good accuracy for displacements as well as for internal moments. The refined element, though of a non-conforming type, compares favorably with most presently known rectangular or quadrilateral finite elements.

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Резюме — Во время последнего десятилетия много усилия было затрачено на определение надежных матричных параметров жесткости конечных элементов различных форм для изгиба пластин. Большое внимание уделяли трехугольным, прямоугольным и четырехугольным элементам. Недавние исследования имеющихся в настоящее время элементов даны Бэллом (спр. 1) и Галлахером (спр. 6). Обзор прямоугольных конечных элементов изгиба пластин даны Глоуом и Точером (спр. 3). Сравнительные изучения показали, что прямоугольные элементы точнее, чем трехугольные при той же степени свободы.

В настоящее время для анализа изгиба применяются несколько прямоугольных и четырехугольных конечных элементов. Большинство элементов показывают хорошую сходимость смещений. Однако степень сходимости сильно различается для различных элементов. Кроме того, несмотря на приемлемую точность смещений, некоторые элементы очень неточные по отношению к внутренним моментам.